Partial Differential Equations - Resit exam

You have 3 hours to complete this exam. Please show all work. The exam consists of 5 questions for a total of 90 points. You get 10 additional points for putting your name and student ID on every page you wish to be graded bringing the total to 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) (20 points) Prove that if $u(x, y)$ is harmonic in a bounded region Ω and u is $C^1(\overline{\Omega})$ then $w = |\nabla u|^2$ attains its maximum on $\partial\Omega$, the boundary of Ω. (Hint, what is the sign of Δw?)

(2) (15 points) Consider the function $f(x) = x$.
  (a) (6 points) Compute the Fourier series of f on the interval $[-\pi, \pi]$.
  (b) (4 points) Draw and write down explicitly to what function the Fourier series converges.
  (c) (5 points) Does it converge pointwise? Does it converge uniformly? Justify the type of convergence by stating the appropriate theorem.

(3) (25 points) Consider the initial value problem

$$\begin{cases} -u'' + \omega^2 u = h(x), & x \in \mathbb{R},\ \omega > 0, \\ u(x) \to 0 \text{ as } |x| \to \infty \end{cases} \tag{1}$$

  (a) (5 points) Give the definition of the free space Green's function for the Poisson problem.
  (b) (10 points) Show that, if you know the free space Green's function G_0, you can compute the solution of (1).
  (c) (10 points) Use the Fourier Transform to directly solve (1) in terms of a convolution.

(4) (10 points) Consider the first order PDE on the Euclidean plane

$$u_x - 3x^2 u_y = 0. \tag{2}$$

  (a) (7 points) Find the general solution of (2).
  (b) (3 points) Find the solution of (2) satisfying $u(0, y) = -y^2$, $y \in \mathbb{R}$.

(5) (20 points) Let D_a be the half disk of radius a defined by

$$D_a = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2,\ y \geq 0\right\}.$$

Consider on D_a the Laplace equation $\Delta u = 0$ with boundary conditions

$$u(a, \theta) = h(\theta), \quad \theta \in [0, \pi],$$
$$u(r, 0) = u(r, \pi) = 0, \quad r \in [0, a].$$

  (a) (5 points) Separate Laplace equation in polar coordinates r, θ. Hint: let $u(r, \theta) = R(r)\Theta(\theta)$ and write the two separated equations, for R and for Θ respectively.
  (b) (7 points) Solve the eigenvalue equation for Θ for the given boundary conditions. Consider known that the problem has no complex eigenvalues, but check for positive, negative, or zero eigenvalues.
  (c) (4 points) Solve the differential equation for R.
  (d) (4 points) Write the general solution $u(r, \theta)$ and use it to compute the solution when $h(\theta) = \sin\theta$.